Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On the Structure of Finite Continuous Groups.

By J. A. Bullard.

Introduction.

The structure of a finite continuous group is determined in part by the "characteristic equation of the group," that is, the characteristic equation of the general infinitesimal transformation of the adjoint group of the given group. It is the purpose of this paper to show how certain properties of a group of linear homogeneous transformations can be obtained at once from the characteristic equation of the general infinitesimal transformation of the group, and thus how the type of structure is in part determined immediately from the infinitesimal transformations of such a group without the determination of the adjoint group. In § 1 I show how to find the structural constants, and thus the roots of the characteristic equation, of the general infinitesimal transformation of the adjoint group of the general linear homogeneous group relative to a given subgroup; in § 2 I find similar results for the special linear homogeneous group relative to a given subgroup. These results lead to certain relations, given in § 3, between the characteristic equation of a group of linear homogeneous transformations and the characteristic equation of the general infinitesimal transformation of the given group. By the aid of these relations I establish theorems I–X by which certain properties of such a group can be at once determined from the general infinitesimal transformation of the group.

Section 1.

Let G denote the general linear homogeneous group in n variables. The symbols of the infinitesimal transformations of G are the n^2 differential operators

$$\varepsilon_{ij}=x_i\frac{\partial}{\partial x_j}, \qquad (i, j=1, 2, \ldots, n),$$

or any n^2 linearly independent linear homogeneous functions of these operators with constant coefficients. Let G_r denote any given subgroup of the general linear homogeneous group with r parameters, the symbols of whose infinitesimal transformations are the r given differential operators

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r, \text{ where } \mathscr{A}_h=\sum_{i, j=1}^{n} a_{hij}\varepsilon_{ij}, \qquad (h=1, 2, \ldots, r).$$

Then
$$\left\| \begin{matrix} a_{111}, a_{112}, \ldots, a_{11n}, a_{121}, \ldots, a_{1n1}, a_{1n2}, \ldots, a_{1nn} \\ a_{211}, a_{212}, \ldots, a_{21n}, a_{221}, \ldots, a_{2n1}, a_{2n2}, \ldots, a_{2nn} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r11}, a_{r12}, \ldots, a_{r1n}, a_{r21}, \ldots, a_{rn1}, a_{rn2}, \ldots, a_{rnn} \end{matrix} \right\| \neq 0,$$

and
$$(\mathcal{A}_i, \mathcal{A}_j) = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i = \sum_{h=1}^{r} c_{jih}\mathcal{A}_h, \qquad (i, j = 1, 2, \ldots, r).$$

We may take as the infinitesimal transformations of G the differential operators $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$, together with any n^2-r other differential operators $\mathcal{A}_{r+1}, \mathcal{A}_{r+2}, \ldots, \mathcal{A}_{n^2}$ defined as follows:

$$\mathcal{A}_{r+h} = \sum_{i,j=1}^{n} a_{r+h,i,j}\varepsilon_{ij}, \qquad (h=1, 2, \ldots, n^2-r),$$

provided these n^2 differential operators are independent, that is, provided the n^2 $\mathcal{A}$'*s* are linearly independent linear homogeneous functions of the ε's. In which case

$$\left| \begin{matrix} a_{i11}, a_{i12}, \ldots, a_{i1n}, a_{i21}, \ldots, a_{in1}, a_{in2}, \ldots, a_{inn} \\ i=1, 2, \ldots, n^2 \end{matrix} \right| \neq 0.$$

I shall employ the following notation. The constituent of any matrix B in the *p-th* row and *q-th* column will be denoted by $(B)_{pq}$ and the determinant of B by $|B|$. Thus, if T denote the above matrix, $|T| \neq 0$ and $(T)_{i,(\mu-1)n+\nu} = a_{i\mu\nu}$, $(i=1, 2, \ldots, n^2;\ \mu, \nu=1, 2, \ldots, n)$. We shall now have, in Cayley's matrical notation,

$$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n^2}) = T(\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{21}, \ldots, \varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}). \quad (1)$$

For $1 \leq i \leq n^2$ the equations of the infinitesimal transformation of G corresponding to $\mathcal{A}_i$, namely,

$$x'_\mu = x_\mu + \delta t\mathcal{A}_i x_\mu, \qquad (\mu=1, 2, \ldots, n),$$

may be written $(x'_1, x'_2, \ldots, x'_n) = (1+\delta t A_i)(x_1, x_2, \ldots, x_n)$ where A_i is the matrix corresponding to $\mathcal{A}_i$ and

$$(A_i)_{\mu\nu} = a_{i\mu\nu}, \qquad (\mu, \nu=1, 2, \ldots, n). \qquad (2)$$

For our present purpose we may substitute the matrices $A_1, A_2, \ldots, A_{n^2}$ for the respective differential operators $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n^2}$, regarding the former as the symbols of the infinitesimal transformations of G. Similarly, if e_{ij}, $(i, j=1, 2, \ldots, n)$, denote the matrix whose constituents are all zero except that in the *i-th* row and *j-th* column which is equal to 1, we may substitute

e_{ij}, $(i, j=1, 2, \ldots, n)$, for ε_{ij}. The general infinitesimal transformation of G_r is then represented by

$$A'=\sum_{h=1}^{r}\alpha_h\mathscr{A}_h=\sum_{h=1}^{r}\sum_{i,j=1}^{n}\alpha_h a_{hij}\varepsilon_{ij}, \text{ or by the matrix } A=\sum_{h=1}^{r}\alpha_h A_h,$$

where $(A')_{\mu\nu}=\sum_{i=1}^{r}\alpha_i a_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$. The general infinitesimal transformation of the general linear homogeneous group is represented by

$$A=\sum_{h=1}^{n^2}\alpha_h\mathscr{A}_h=\sum_{h=1}^{n^2}\sum_{i,j=1}^{n}\alpha_h a_{hij}\varepsilon_{ij}, \text{ or by the matrix } A=\sum_{h=1}^{n^2}\alpha_h A_h,$$

where $(A)_{\mu\nu}=a_{\mu\nu}=\sum_{i=1}^{n^2}\alpha_i a_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$.

We shall have then

$$(A_i, A_j)=\sum_{k=1}^{r}c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, r), \tag{3}$$

$$(A_i, A_j)=\sum_{k=1}^{n^2}c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, n^2),$$

where c_{ijk}, $(i, j, k=1, 2, \ldots, n^2)$, are the structural constants of G, and c_{ijk}, $(i, j, k=1, 2, \ldots, r)$, are the structural constants of G_r. Since there is no linear relation with constant coefficients between $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r$, the corresponding matrices $A_1, A_2, \ldots, A_r$ are linearly independent; similarly, since there is no linear relation with constant coefficients between $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n^2}$, then $A_1, A_2, \ldots, A_{n^2}$ are linearly independent.

Let Γ denote the adjoint of the general linear homogeneous group G, and Γ_r the adjoint of G_r. The symbols of the infinitesimal transformations of the adjoint are differential operators which may be replaced for our present purpose by the matrices representing the square arrays of their coefficients. Thus, if $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n^2}$ are chosen as the symbols of the infinitesimal transformations of G, the corresponding symbols of the infinitesimal transformations of the adjoint are $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{n^2}$, where $\mathscr{E}_i=\sum_{j,k=1}^{n^2}c_{ijk}\alpha_j\frac{\partial}{\partial\alpha_k}$, $(i=1, 2, \ldots, n^2)$. The equations, $\alpha'_\mu=\alpha_\mu+\delta t\mathscr{E}_i\alpha_\mu=\alpha_\mu+\delta t\sum_{j=1}^{n^2}c_{ij\mu}\alpha_j$, $(\mu=1, 2, \ldots, n^2)$, defining the infinitesimal transformation of Γ may be written in matrical form as follows: $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n^2})=(1+\delta tE_i)(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$, where E_i is the matrix defined by the equations

$$(E_i)_{\nu\mu}=c_{i\mu\nu}, \qquad (i, \mu, \nu=1, 2, \ldots, n^2); \tag{4}$$

and we may substitute for the symbols $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{n^2}$, the respectively corresponding matrices $E_1, E_2, \ldots, E_{n^2}$. A similar remark applies to the group Γ_r, whose symbols of infinitesimal transformation are $\mathcal{E}'_1, \mathcal{E}'_2, \ldots, \mathcal{E}'_r$, where $\mathcal{E}'_i = \sum_{j,k=1}^{r} c_{ijk}\alpha_j \frac{\partial}{\partial \alpha_k}$, $(i=1, 2, \ldots, r)$, and these may be replaced by the respectively corresponding matrices $E'_1, E'_2, \ldots, E'_r$, where

$$(E'_i)_{\nu\mu} = c_{i\mu\nu}, \qquad (i, \mu, \nu = 1, 2, \ldots, r). \tag{5}$$

We have

$$(E_i, E_j) = \sum_{k=1}^{n^2} c_{ijk} E_k, \qquad (i, j = 1, 2, \ldots, n^2),$$

$$(E'_i, E'_j) = \sum_{k=1}^{r} c_{ijk} E'_k, \qquad (i, j = 1, 2, \ldots, r).$$

When $A_1, A_2, \ldots, A_{n^2}$ are chosen as the symbols of the infinitesimal transformations of G, the determination of the structural constants c_{ijk}, and hence of the matrices $E_1, E_2, \ldots, E_{n^2}$, may be accomplished as follows. The matrices $A_1, A_2, \ldots, A_{n^2}$, constitute a hypercomplex number system. Let $A_i A_j = \sum_{k=1}^{n^2} \gamma_{ijk} A_k$; by a theorem of Lie's (*Transformationsgruppen*, Vol. III, § 140)

$$c_{ijk} = \gamma_{ijk} - \gamma_{jik}. \tag{6}$$

To find γ_{ijk} we note that $e_{\mu\nu}$, $(\mu, \nu = 1, 2, \ldots, n)$, constitute a number system equivalent to $A_1, A_2, \ldots, A_{n^2}$, with a multiplication table determined by the equations $e_{\mu\nu} e_{\nu\sigma} = e_{\mu\sigma}$ and $e_{\mu\nu} e_{\lambda\sigma} = 0$, $(\mu, \nu, \lambda, \sigma = 1, 2, \ldots, n;\ \lambda \neq \nu)$. We may put $e_{\mu\nu} e_{\lambda\sigma} = \sum_{\rho,\tau=1}^{n} g_{\mu,\nu,\lambda,\sigma,\rho,\tau}\, e_{\rho\tau}$, provided $g_{\mu,\nu,\lambda,\sigma,\rho,\tau} = 0$ for all values of the subscripts except that

$$g_{\mu,\nu,\nu,\sigma,\mu,\sigma} = 1, \qquad (\mu, \nu, \sigma = 1, 2, \ldots, n).$$

From equation (1), or rather the equation

$$(A_1, A_2, \ldots, A_{n^2}) = T(e_{11}, e_{12}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nn}),$$

we derive

$$(e_{11}, e_{12}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nn}) = T^{-1}(A_1, A_2, \ldots, A_{n^2}). \tag{7}$$

Then $(T)_{i,\,(\mu-1)n+\nu} = a_{i\mu\nu}$,

$$(T^{-1})_{(\mu-1)n+\nu,\,i} = (-1)^{(\mu-1)n+\nu+i} \frac{1}{|T|} \frac{\partial}{\partial a_{i\mu\nu}} |T|,$$

$$(i = 1, 2, \ldots, n^2;\ \mu, \nu = 1, 2, \ldots, n);$$

and equations (1) and (7) are respectively equivalent to the two systems of equations

$$A_i = \sum_{\mu,\nu=1}^{n} a_{i\mu\nu} e_{\mu\nu} = \sum_{\mu,\nu=1}^{n} (T)_{i,\,(\mu-1)n+\nu} e_{\mu\nu}, \qquad (i = 1, 2, \ldots, n^2),$$

$$e_{\mu\nu} = \sum_{i=1}^{n^2} (T^{-1})_{(\mu-1)n+\nu,\,i} A_i, \qquad (\mu, \nu = 1, 2, \ldots, n).$$

Therefore,
$$\begin{aligned}A_iA_j &= \sum_{\mu,\nu=1}^{n}(T)_{i,(\mu-1)n+\nu}e_{\mu\nu}\sum_{\lambda,\sigma=1}^{n}(T)_{j,(\lambda-1)n+\sigma}e_{\lambda\sigma}\\ &= \sum_{\mu,\nu,\lambda,\sigma=1}^{n}(T)_{i,(\mu-1)n+\nu}(T)_{j,(\lambda-1)n+\sigma}e_{\mu\nu}e_{\lambda\sigma}\\ &= \sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{i,(\mu-1)n+\nu}(T)_{j,(\lambda-1)n+\sigma}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}e_{\rho,\tau}\\ &= \sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{i,(\mu-1)n+\nu}(T)_{j,(\lambda-1)n+\sigma}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}\sum_{k=1}^{n^2}(T^{-1})_{(\rho-1)n+\tau,k}A_k,\\ &\qquad (i,j=1,2,\ldots\ldots,n^2).\end{aligned}$$

If the matrix M_i, $(1\leqq i\leqq n^2)$ of order n^2 be defined by the equations
$$(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\nu=1}^{n}(T)_{i,(\mu-1)n+\nu}g_{\mu,\nu,\lambda,\sigma,\rho,\tau},\quad (\lambda,\sigma,\rho,\tau=1,2,\ldots\ldots,n),\tag{8}$$
we may write
$$\begin{aligned}A_iA_j &= \sum_{k=1}^{n^2}\sum_{\rho,\tau,\lambda,\sigma=1}^{n}(T)_{j,(\lambda-1)n+\sigma}(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(T^{-1})_{(\rho-1)n+\tau,k}A_k\\ &= \sum_{k=1}^{n^2}\sum_{\lambda,\sigma,\rho,\tau=1}^{n}(\breve{T}^{-1})_{k,(\rho-1)n+\tau}(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(\breve{T})_{(\lambda-1)n+\sigma,j}A_k\\ &= \sum_{k=1}^{n^2}(\breve{T}^{-1}M_i\breve{T})_{kj}A_k,\qquad (i,j=1,2,\ldots\ldots,n^2),\end{aligned}\tag{9}$$

where $\breve{T}$ and $\breve{T}^{-1}$ denote respectively the transverse or conjugate of T and T^{-1}.* Therefore, since the A's are linearly independent,
$$\gamma_{ijk}=(\breve{T}^{-1}N_i\breve{T})_{kj},\qquad (i,j,k=1,2,\ldots\ldots,n^2).\tag{10}$$
Interchanging A_i and A_j in (9) we obtain
$$A_jA_i=\sum_{k=1}^{n^2}\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{j,(\lambda-1)n+\sigma}(T)_{i,(\mu-1)n+\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau}(T^{-1})_{(\rho-1)n+\tau,j}A_k,\qquad (i,j=1,2,\ldots\ldots,n^2).$$
and defining the matrix N_i by the equations
$$(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\nu=1}^{n}(T)_{i,(\mu-1)n+\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau},\qquad (\lambda,\sigma,\rho,\tau=1,2,\ldots\ldots,n;\ i=1,2,\ldots\ldots,n^2),\tag{11}$$
$$\begin{aligned}A_jA_i &= \sum_{k=1}^{n^2}\sum_{\lambda,\sigma,\rho,\tau=1}^{n}(T)_{j,(\lambda-1)n+\sigma}(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(T^{-1})_{(\rho-1)n+\tau,k}A_k\\ &= \sum_{k=1}^{n^2}(\breve{T}^{-1}N_i\breve{T})_{kj}A_k,\qquad (i,j=1,2,\ldots\ldots,n^2),\end{aligned}$$

and therefore,
$$\gamma_{jik}=(\breve{T}^{-1}N_i\breve{T})_{kj},\qquad (i,j,k=1,2,\ldots\ldots,n^2).\tag{12}$$

---

* The transverse of any matrix B is obtained by interchanging rows and columns so that $(B)_{ij}=b_{ij}=(\breve{B})_{ji}$. We have $(\breve{B})^{-1}=\breve{(B^{-1})}$; and if C is any second matrix of order n, $\breve{(B\pm C)}=\breve{B}\pm\breve{C}$ and $\breve{(BC)}=(\breve{C}\breve{B})$.

By virtue of the equations (6), (11), (12), we now have

$$c_{ijk}=(\breve{T}^{-1}M_i\breve{T})_{kj}-(\breve{T}^{-1}N_i\breve{T})_{kj}=(\breve{T}^{-1}(M_i-N_i)\breve{T})_{kj}, \qquad (i,\ j,\ k=1,\ 2,\ \ldots,\ n^2),$$

and therefore, by equation (4) which defines the matrices $E_1, E_2, \ldots, E_{n^2}$,

$$E_i=\breve{T}^{-1}(M_i-N_i)\breve{T}, \qquad (i=1,\ 2,\ \ldots,\ n^2). \tag{13}$$

The matrices M_i and N_i are respectively defined by the equations

$$(M_i)_{(\rho-1)n+\tau,\ (\lambda-1)n+\sigma}=\sum_{\mu,\ \nu=1}^{n} a_{i\mu\nu}g_{\mu,\ \nu,\ \lambda,\ \sigma,\ \rho,\ \tau}=\sum_{\mu,\ \nu=1}^{n}(A_i)_{\mu\nu}g_{\mu,\ \nu,\ \lambda,\ \sigma,\ \rho,\ \tau}, \tag{8}$$

$$(N_i)_{(\rho-1)n+\tau,\ (\lambda-1)n+\sigma}=\sum_{\mu,\ \nu=1}^{n} a_{i\mu\nu}g_{\lambda,\ \sigma,\ \mu,\ \nu,\ \rho,\ \tau}=\sum_{\mu,\ \nu=1}^{n}(A_i)_{\mu\nu}g_{\lambda,\ \sigma,\ \mu,\ \nu,\ \rho,\ \tau}, \tag{11}$$

$$(i=1,\ 2,\ \ldots,\ n^2),$$

where $g_{\mu,\ \nu,\ \lambda,\ \sigma,\ \rho,\ \tau}=0$ for all values of the subscripts except that $g_{\mu,\ \nu,\ \nu,\ \sigma,\ \mu,\ \sigma}=1$, $(\mu,\ \nu,\ \sigma=1,\ 2,\ \ldots,\ n)$. Whence it follows that

$$(M_i)_{(\mu-1)n+\sigma,\ (\nu-1)n+\sigma}=a_{i\mu\nu}=(A_i)_{\mu\nu}, \qquad (\mu,\ \nu,\ \sigma=1,\ 2,\ \ldots,\ n),$$

$$(N_i)_{(\lambda-1)n+\nu,\ (\lambda-1)n+\mu}=a_{i\mu\nu}=(A_i)_{\mu\nu}, \qquad (\lambda,\ \mu,\ \nu=1,\ 2,\ \ldots,\ n),$$

and that all other constituents of M_i and N_i $(1 \leqq i \leqq n^2)$ are zero. Hence we find that the matrix M_i which is of order n^2, may be represented in the following way: $(M_i)_{\mu\nu}=M_{i\mu\nu}$, $(\mu,\ \nu=1,\ 2,\ \ldots,\ n)$, where, for $1 \leqq i \leqq n^2$, the constituent $M_{i\mu\nu}$ is itself a matrix of order n defined as follows: $(M_{i\mu\nu})_{pp}=a_{i\mu\nu}$, $(M_{i\mu\nu})_{pq}=0$, $(p,\ q=1,\ 2,\ \ldots,\ n;\ p \neq q)$. If I denotes the identical substitution in n variables we may write

$$(M_i)_{\mu\nu}=(A_i)_{\mu\nu}I, \qquad (\mu,\ \nu=1,\ 2,\ \ldots,\ n;\ i=1,\ 2,\ \ldots,\ n^2). \tag{14}$$

Similarly, we find that N_i can be expressed in terms of constituent matrices as follows:

$$(N_i)_{pp}=\breve{A}_i,\ (N_i)_{pq}=0, \qquad (p,\ q=1,\ 2,\ \ldots,\ n;\ p \neq q;\ i=1,\ 2,\ \ldots,\ n^2). \tag{15}$$

If B, C, Q and R are any four matrices of order n, for two matrices W_b and W_c such that $(W_b)_{\mu\nu}=(B)_{\mu\nu}Q$ and $(W_c)_{\mu\nu}=(C)_{\mu\nu}R$, $(\mu,\ \nu=1,\ 2,\ \ldots,\ n)$, we have

$$(W_bW_c)_{\mu\nu}=\sum_{h=1}^{n}(B)_{\mu h}(C)_{h\nu}QR^{*}=(BC)_{\mu\nu}QR=(W_{bc})_{\mu\nu}, \quad (\mu,\nu=1,2,\ldots,n). \tag{16}$$

By this formula we obtain

$$(M_iM_j)_{\mu\nu}=(A_iA_j)_{\mu\nu}I, \tag{17}$$

$$(M_i^2)_{\mu\nu}=(A_i^2)_{\mu\nu}I, \quad (M_i^3)_{\mu\nu}=(M_iM_i^2)_{\mu\nu}=(A_i^3)_{\mu\nu}I, \tag{18}$$

and for any positive integer p

$$(M_i^p)_{\mu\nu}=(A_i^p)_{\mu\nu}I, \qquad (\mu,\ \nu=1,\ 2,\ \ldots,\ n;\ i,\ j=1,\ 2,\ \ldots,\ n^2)\ ; \tag{19}$$

---

* The law of multiplication for matrices whose constituents are themselves matrices is the same as for matrices whose constituents are scalars.

again from (15) we have,

$$\left.\begin{aligned} (N_iN_j)_{\mu\nu} &= \sum_{h=1}^{n}(N_i)_{\mu h}(N_j)_{h\nu}=0, \\ (N_iN_j)_{\mu\mu} &= \sum_{h=1}^{n}(N_i)_{\mu h}(N_j)_{h\mu}=(N_i)_{\mu\mu}(N_j)_{\mu\mu}=\breve{A}_i\breve{A}_j, \end{aligned}\right\} \quad (20)$$

and for any positive integer p

$$(N_i^p)_{\mu\mu}=\breve{A}_i^p, \quad (N_i^p)_{\mu\nu}=0, \qquad (\mu, \nu=1, 2, \ldots, n;\ \mu\neq\nu;\ i, j=1, 2, \ldots, n^2). \quad (21)$$

Further, by equations (16), (19) and (21) for positive integers p, q, r, s, t, v we obtain

$$\begin{aligned} (M_i^sN_j^p)_{\mu\nu} &= \sum_{h=1}^{n}(M_i^s)_{\mu h}(N_j^p)_{h\nu}=(M_i^s)_{\mu\nu}(N_j^p)_{\nu\nu}=(A_i^s)_{\mu\nu}I\breve{A}_j^p=(N_j^p)_{\mu\mu}(M_i^s)_{\mu\nu} \\ &= \sum_{h=1}^{n}(N_j^p)_{\mu h}(M_i^s)_{h\nu}=(N_j^pM_i^s)_{\mu\nu}, \end{aligned} \quad (22)$$

$$(M_i^pN_i^qM_j^rN_j^s)_{\mu\nu}=((M_i^pN_i^q)(M_j^rN_j^s))_{\mu\nu}=(A_i^pA_j^r)_{\mu\nu}\breve{A}_i^q\breve{A}_j^s, \quad (23)$$

$$(M_i^pN_i^qM_j^rN_j^sM_k^tN_k^v)_{\mu\nu}=((M_i^pN_i^qM_j^rN_j^s)(M_k^tN_k^v))_{\mu\nu}=(A_i^pA_j^rA_k^t)_{\mu\nu}\breve{A}_i^q\breve{A}_j^s\breve{A}_k^v,$$
$$(\mu, \nu=1, 2, \ldots, n;\ i, j, k=1, 2, \ldots, n^2). \quad (24)$$

I shall employ the symbol S prefixed to a matrix M whose constituents are scalars, designating SM as the *scalar* of M, to denote the sum of the constituents in the principal diagonal of M. Then, if A and B are any two matrices,

$$S(A\pm B)=SA\pm SB, \quad SAB=SBA, \quad S\breve{A}=SA. \quad (25)$$

Further, if ρ is any scalar

$$S\rho A=\rho SA; \quad (26)$$

and if I is the identical substitution in n variables,

$$SI=n. \quad (27)$$

Moreover, SM is the sum of the roots of the characteristic equation of M, $|\omega-M|=0$, and SM^p is the sum of the *p-th* powers of the roots. In the case of a matrix $(M)_{\mu\nu}=M_{\mu\nu}$, $(\mu, \nu=1, 2, \ldots)$, whose constituents, the *M's* with double subscripts, are themselves matrices, those in the principal diagonal being square matrices, I shall define SM as follows:

$$SM=SM_{11}+SM_{22}+\ldots.$$

It may be readily seen that the properties of the scalar function given in equations (25), (26) and (27) hold in this case also.

We then have from equation (13), $SE_i = S\breve{T}^{-1}(M_i - N_i)\breve{T}$, and since the scalar product of two or more matrices is unchanged by an interchange of the factors leaving the cyclic order unaltered,

$$SE_i = S(M_i - N_i) = SM_i - SN_i, \qquad (i=1, 2, \ldots, n^2).$$

By equation (14), we have $SM_i = \sum_{\mu=1}^{n} (A_i)_{\mu\mu} SI = SA_i \cdot SI = nSA_i$, and by equation (15), $SN_i = nS\breve{A}_i = nSA_i$. Substituting these results in the above we obtain

$$SE_i = 0, \qquad (i=1, 2, \ldots, n^2). \tag{28}$$

Also

$$\begin{aligned} SE_iE_j &= S\breve{T}^{-1}(M_i - N_i)\breve{T} \cdot \breve{T}^{-1}(M_j - N_j)\breve{T} = S(M_i - N_i)(M_j - N_j) \\ &= SM_iM_j - SM_iN_j - SN_iM_j + SN_iN_j, \qquad (i, j=1, 2, \ldots, n^2). \end{aligned} \tag{29}$$

From equation (17) we obtain $SM_iM_j = \sum_{\mu=1}^{n} (A_iA_j)_{\mu\mu} SI = SA_iA_jSI = nSA_iA_j$; from equation (20), $SN_iN_j = nS\breve{A}_i\breve{A}_j = nSA_iA_j$; from equation (22), $SM_iN_j = SN_jM_i = \sum_{\mu=1}^{n} (A_i)_{\mu\mu} S\breve{A}_j = SA_iS\breve{A}_j = SA_iSA_j$, and substituting these values in equation (29) we find

$$SE_iE_j = 2(nSA_iA_j - SA_iSA_j), \qquad (i, j=1, 2, \ldots, n^2). \tag{30}$$

For any positive integer h we have

$$SE_i^h = (S\breve{T}^{-1}(M_i - N_i)\breve{T})^h = S(M_i - N_i)^h = \sum_{\mu=0}^{h} (-1)^\mu \binom{h}{\mu} SM_i^{h-\mu}N_i^\mu, \qquad (i=1, 2, \ldots, n^2).$$

Since for any odd integer, $h = 2p+1$, the number of terms in the expansion is even and the even terms have a negative sign, and the binomial coefficients $\binom{h}{j}$ and $\binom{h}{h-j}$ are equal, we may write

$$\begin{aligned} SE_i^{2p+1} &= \sum_{\mu=0}^{2p+1} (-1)^\mu \binom{2p+1}{\mu} SM_i^{2p-\mu+1}N_i^\mu \\ &= \sum_{\mu=0}^{p} (-1)^\mu \binom{2p+1}{\mu} (SM_i^{2p-\mu+1}N_i^\mu - SM_i^\mu N_i^{2p-\mu+1}), \\ &\qquad (i=1, 2, \ldots, n^2). \end{aligned} \tag{31}$$

From equations (19), (21) and (22), respectively, we obtain

$$SM_i^q = \sum_{\mu=1}^{n} (A_i^q)_{\mu\mu} SI = SA_i^qSI = nSA_i^q,$$

$$SN_i^q = nS\breve{A}_i^q = nSA_i^q, \quad SM_i^qN_i^s = \sum_{\mu=1}^{n} (A_i^q)_{\mu\mu} S\breve{A}_i^s = SA_i^qSA_i^s = SM_i^sN_i^q,$$

and substituting in equation (31)

$$SE_i^{2p+1}=0, \qquad (i=1,\, 2,\, \ldots.,\, n^2). \tag{32}$$

For any even integer, $h=2p$, we have

$$SE_i^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SA_i^{2p-\mu}SA_i^{\mu}, \qquad (i=1,\, 2,\, \ldots.,\, n^2), \tag{33}$$

where SA_i^0 is assigned the value n. For any positive integers p and q

$$\begin{aligned} SE_i^pE_j^q&=S(\breve{T}^{-1}(M_i-N_i)\breve{T})^p(\breve{T}^{-1}(M_j-N_j)\breve{T})^q=S(M_i-N_i)^p(M_j-N_j)^q \\ &=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}(-1)^{\mu+\nu}\binom{p}{\mu}\binom{q}{\nu}SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}, \quad (i,\, j=1,\, 2,\, \ldots.,\, n^2). \end{aligned} \tag{34}$$

From (23) we obtain $SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}=SA_i^{p-\mu}A_j^{q-\nu}S\breve{A}_i^{\mu}\breve{A}_j^{\nu}=SA_i^{p-\mu}A_j^{q-\nu}SA_i^{\mu}A_j^{\nu}$, and substituting in (34),

$$SE_i^pE_j^q=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}(-1)^{\mu+\nu}\binom{p}{\mu}\binom{q}{\nu}SA_i^{p-\mu}A_j^{q-\nu}SA_i^{\mu}A_j^{\nu}, \qquad (i,\, j=1,\, 2,\, \ldots.,\, n^2), \tag{35}$$

where $A_i^0=A_j^0=I$. Again, for positive integers p, q and r

$$\begin{aligned} SE_i^pE_j^qE_k^r&=S(M_i-N_i)^p(M_j-N_j)^q(M_k-N_k)^r \\ &=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}\sum_{\sigma=0}^{r}(-1)^{\mu+\nu+\sigma}\binom{p}{\mu}\binom{q}{\nu}\binom{r}{\sigma}SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}M_k^{r-\sigma}N_k^{\sigma}, \\ &\qquad (i,\, j,\, k=1,\, 2,\, \ldots.,\, n^2). \end{aligned} \tag{36}$$

From equation (24) $SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}M_k^{r-\sigma}N_k^{\sigma}=SA_i^{p-\mu}A_j^{q-\nu}A_k^{r-\sigma}S\breve{A}_i^{\mu}\breve{A}_j^{\nu}\breve{A}_k^{\sigma}$, and substituting in (36) we have

$$\begin{aligned} SE_i^pE_j^qE_k^r=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}\sum_{\sigma=0}^{r}(-1)^{\mu+\nu+\sigma}\binom{p}{\mu}\binom{q}{\nu}\binom{r}{\sigma}SA_i^{p-\mu}A_j^{q-\nu}A_k^{r-\sigma}S\breve{A}_i^{\mu}\breve{A}_j^{\nu}\breve{A}_k^{\sigma}, \\ (i,\, j,\, k=1,\, 2,\, \ldots.,\, n^2), \end{aligned} \tag{37}$$

where $A_i^0=A_j^0=A_k^0=I$. In particular

$$\begin{aligned} SE_iE_jE_k=n(SA_iA_jA_k-S\breve{A}_i\breve{A}_j\breve{A}_k)=nS(A_i,\, A_j)A_k, \\ (i,\, j,\, k=1,\, 2,\, \ldots.,\, n^2). \end{aligned} \tag{38}$$

The equations (32) and (33) enable us to establish certain relations between the roots of the characteristic equation,

$$|\omega-E_i|=\omega^{n^2}-\Psi_1\omega^{n^2-1}+\Psi_2\omega^{n^2-2}-\ldots.\pm\Psi_{n^2-1}\omega\mp\Psi_{n^2}=0, \quad (\Psi_{n^2}=0),^*$$

of E_i and the roots of the characteristic equation,

$$|\rho-A_i|=\rho^n-P_1\rho^{n-1}+P_2\rho^{n-2}-\ldots.\mp P_{n-1}\rho\pm P_n=0,$$

of A_i. Let $\omega_{11},\, \omega_{12},\, \ldots.,\, \omega_{nn}$ denote the roots of the equation $|\omega-E_i|=0$, and $\mathscr{S}_q$ the sum of the *q-th* powers of the roots; let $\rho_1,\, \rho_2,\, \ldots.,\, \rho_n$ denote the roots of $|\rho - A_i| = 0$, and s_q the sum of their *q-th* powers. By equation (32) $\mathscr{S}_{2p+1} = SE_i^{2p+1} = 0$, $(p=0, 1, 2, \ldots.)$, wherefore $\Psi_{2p+1}=0$, $(p=0, 1, 2, \ldots.)$, and the non-zero roots of $|\omega - E_i| = 0$ occur in pairs of equal value but opposite sign;* from equation (33)

---

* Killing has shown that the characteristic equation of the general infinitesimal transformation of the adjoint has at least one zero root (*Math. Ann.*, Vol. XXXI, p. 260).

$$\mathscr{S}_{2p} = \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} s_{2p-\mu} s_{\mu}, \qquad (p=0, 1, 2, \ldots.),$$

from which relation it then follows that the non-zero roots of $|\omega - E_i| = 0$ not exceeding $n^2 - n$ in number, shall be related to the roots of $|\rho - A_i| = 0$ by the equation $\omega_{ij} = \rho_i - \rho_j$, $(i \neq j;\ i, j = 1, 2, \ldots., n)$.†

For any linear function of the *E's* with constant coefficients, $\sum_{i=1}^{n^2} \alpha_i E_i$, we have

$$E = \sum_{i=1}^{n^2} \alpha_i E_i = \breve{T}^{-1}(M-N)\breve{T}, \tag{39}$$

where $M = \sum_{i=1}^{n^2} \alpha_i M_i$ and $N = \sum_{i=1}^{n^2} \alpha_i N_i$. If $A = \sum_{i=1}^{n^2} \alpha_i A_i$ we see from equation (14) that $(M)_{\mu\nu} = (A)_{\mu\nu} I$, $(\mu, \nu = 1, 2, \ldots., n)$, and by the aid of equation (16) we can show that for any positive integer p, $(M^p)_{\mu\nu} = (A^p)_{\mu\nu} I$, $(\mu, \nu = 1, 2, \ldots., n)$. It is evident from equation (15) that $(N)_{\mu\mu} = \breve{A}$ and $(N^p)_{\mu\mu} = \breve{A}^p$ for any positive integer p $(\mu = 1, 2, \ldots., n)$ and that all other constituent matrices of these matrices are zero. Also corresponding to equation (22) we have $(M^p N^q)_{\mu\nu} = (A^p)_{\mu\nu} \breve{A}^q = (N^q M^p)_{\mu\nu}$, $(\mu, \nu = 1, 2, \ldots., n)$ for any positive integers p and q. Hence

$$\left.\begin{aligned} SM^p &= \sum_{\mu=1}^{n} (A^p)_{\mu\mu} SI = nSA^p, \quad SN^p = \sum_{\mu=1}^{n} (N^p)_{\mu\mu} = nSA^p, \\ SM^p N^q &= \sum_{\mu=1}^{n} (A^p)_{\mu\mu} S\breve{A}^q = SA^p SA^q = SM^q N^p. \end{aligned}\right\} \tag{40}$$

---

* Substituting these values of $\mathscr{S}_{2p+1}$ in Newton's formula,

$$\mathscr{S}_k - \Psi_1 \mathscr{S}_{k-1} + \Psi_2 \mathscr{S}_{k-2} - \ldots. + (-1)^{k-1} \Psi_{k-1} \mathscr{S}_1 + (-1)^k k \Psi_k = 0,$$

and considering only $k = 1, 3, 5, \ldots.$, we obtain $\Psi_{2p+1} = 0$ for $p = 0, 1, 2, \ldots.$.

† Let $\mathscr{S}_h$ be the sum of the *h-th* powers of the $n(n-1)$ non-zero roots $\rho_i - \rho_j$, $(i, j = 1, 2, \ldots., n;\ i \neq j)$. Each root is paired with its negative and hence $\mathscr{S}_{2p+1} = 0$, $(p = 1, 2, \ldots.)$. Let

$$\phi(\rho) = \sum_{i=1}^{n} (\rho - \rho_i)^{2p} = \sum_{i=1}^{n} \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} \rho^{2p-\mu} \rho_i^{\mu} = \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} \rho^{2p-\mu} s_{\mu}, \quad (p = 1, 2, \ldots.).$$

Then $$\mathscr{S}_{2p} = \sum_{j=1}^{n} \phi(\rho_j) = \sum_{j=1}^{n} \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} \rho_j^{2p-\mu} s_{\mu} = \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} s_{2p-\mu} s_{\mu}, \quad (p = 1, 2, \ldots.).$$

From equation (39) we obtain

$$\begin{aligned} SE^{2p+1} &= S(\breve{T}^{-1}(M-N)\breve{T})^{2p+1} = S(M-N)^{2p+1} \\ &= \sum_{\mu=0}^{2p+1}(-1)^{\mu}\binom{2p+1}{\mu}SM^{2p-\mu+1}N^{\mu} \\ &= \sum_{\mu=0}^{p}(-1)^{\mu}\binom{2p+1}{\mu}(SM^{2p-\mu+1}N^{\mu}-SM^{\mu}N^{2p-\mu+1}), \quad (p=0, 1, 2, \ldots\ldots), \end{aligned}$$

and $SE^{2p}=S(M-N)^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SM^{2p-\mu}N^{\mu}$, $(p=1, 2, \ldots\ldots)$, and substituting from equations (40) we have

$$SE^{2p+1}=0, \quad SE^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SA^{2p-\mu}SA^{\mu}, \qquad (p=1, 2, \ldots\ldots). \tag{41}$$

The characteristic equation of E, where $E=\sum_{i=1}^{n^2}\alpha_i E_i$ for an arbitrary choice of the α's, is termed by Killing the "characteristic equation" of the group G. The characteristic equation of $\sum_{i=1}^{r}\alpha_i E_i$, for an arbitrary choice of the α's, is termed by Cartan the "characteristic equation" of G relative to the subgroup G_r. It follows from equations (41) that, for $1 \overset{=}{<} r \overset{=}{<} n^2$, the roots of $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ are the n^2 differences of the roots of the equation $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$, and hence we have the following theorem:

The roots of the characteristic equation of the general linear homogeneous group relative to any given subgroup can be obtained by taking all possible differences of the roots of the characteristic equation of the general infinitesimal transformation of this subgroup of the general linear homogeneous group. The non-zero roots do not exceed n^2-n in number and occur in pairs of equal absolute value but opposite sign.

Section 2.

The special linear homogeneous group in n variables, denoted in what follows by G', is constituted by the aggregate of transformations with determinant $+1$ of the general linear homogeneous group, G, in n variables. I shall now assume that G_r is also a subgroup of the special linear homogeneous group, G'. For $1 \leqq i \leqq r$ let $\rho_1, \rho_2, \ldots\ldots, \rho_n$ be the roots of the characteristic equation, $|\rho-A_i|=0$, of A_i. The finite transformations of G' are given by e^{tA_i} for all possible values of t. The roots of the characteristic equation of e^{tA_i} are $e^{t\rho_1}, e^{t\rho_2}, \ldots\ldots, e^{t\rho_n}$; and since $e^{t(\rho_1+\rho_2+\ldots\ldots+\rho_n)}=|e^{tA_i}|=1$ for all values of t, $\rho_1+\rho_2+\ldots\ldots+\rho_n=0$. Therefore, $SA_i=0$, $(i=1, 2, \ldots\ldots, r)$. Therefore, $S\sum_{i=1}^{r}a_iA_i=\sum_{i=1}^{r}a_iSA_i=0$.

We may take as the symbols of the infinitesimal transformations of G', any n^2-1 linearly independent linear homogeneous functions of the differential operators ε_{ij}, $(i, j=1, 2, \ldots, n;\ i \neq j)$, and $\varepsilon_{ii}-\varepsilon_{nn}$, $(i=1, 2, \ldots, n-1)$, or of the aggregate of respectively corresponding matrices. Thus we may take as the symbols of the infinitesimal transformations of G', $A_1, A_2, \ldots, A_r$ together with any n^2-r-1 other matrices linear in those above mentioned, the n^2-1 A's defined by equation (1), being linearly independent and subject to the condition $\sum_{\mu=1}^{n} a_{i\mu\mu}=SA_i=0$, $(i=1, 2, \ldots, n^2-1)$. For the symbols of the infinitesimal transformations of G we may take the n^2-1 linearly independent linear homogeneous functions of ε_{ij}, $(i, j=1, 2, \ldots, n;\ i \neq j)$, and $\varepsilon_{ii}-\varepsilon_{nn}$, $(i=1, 2, \ldots, n-1)$, which constitute the symbols of the infinitesimal transformations of G' together with the operator $\sum_{\mu=1}^{n} \varepsilon_{\mu\mu}$, or the aggregate of matrices corresponding to these; and therefore, we may choose as the aggregate of matrices representing the infinitesimal transformations of G the matrices $A_1, A_2, \ldots, A_{n^2-1}$ and $A_{n^2}=\sum_{\mu=1}^{n} e_{\mu\mu}=I$.

Since G' is an invariant subgroup of G, we have

$$(A_i, A_j)=\sum_{k=1}^{n^2-1} c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, n^2-1);$$

moreover, $(A_i, A_{n^2})=0$, $(i=1, 2, \ldots, n^2)$, and therefore,

$$c_{ijk}=0, \qquad (i, j \text{ or } k=n^2). \tag{42}$$

Let Γ' denote the adjoint group of the special linear homogeneous group and let the symbols of the infinitesimal transformations be the differential operators $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{n^2-1}$ where

$$\mathcal{H}_i=\sum_{j=1}^{n^2-1} c_{ijk}\alpha_j \frac{\partial}{\partial \alpha_k}, \qquad (i=1, 2, \ldots, n^2-1),$$

or the respectively corresponding matrices $H_1, H_2, \ldots, H_{n^2-1}$ where

$$(H_i)_{\nu\mu}=c_{i\mu\nu}, \qquad (\mu, \nu, i=1, 2, \ldots, n^2-1). \tag{43}$$

With the infinitesimal transformations of G chosen as above, we see from (4), (42) and (43) that the aggregate of matrices representing the infinitesimal transformations of Γ have the following form:

$$E_i=\begin{vmatrix} H_i & 0 \\ 0 & 0 \end{vmatrix}, \quad E_{n^2}=\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}, \qquad (i=1, 2, \ldots, n^2-1). \tag{44}$$

Then since Γ and Γ' have the same structure as G and G', respectively,

$$(H_i, H_j) = \sum_{k=1}^{n^2-1} c_{ijk} H_k, \quad (E_i, E_j) = \sum_{k=1}^{n^2-1} c_{ijk} E_k, \qquad (i, j = 1, 2, \ldots, n^2-1),$$

$$(E_i, E_{n^2}) = 0, \qquad (i = 1, 2, \ldots, n^2).$$

By the aid of equation (44) we obtain the relations:

$$SE_i = SH_i, \quad SE_iE_j = SH_iH_j, \qquad (i, j = 1, 2, \ldots, n^2-1), \tag{45}$$

and for any natural number p

$$SE_i^p = SH_i^p, \qquad (i = 1, 2, \ldots, n^2-1). \tag{46}$$

It now follows from equations (28) and (45) that

$$SH_i = 0, \qquad (i = 1, 2, \ldots, n^2-1);$$

and since $SA_i = 0$, $(i = 1, 2, \ldots, n^2-1)$, we obtain from equations (30) and (45)

$$SH_iH_j = 2nSA_iA_j, \qquad (i, j = 1, 2, \ldots, n^2-1). \tag{47}$$

Further, we obtain for $p = 0, 1, 2, \ldots$ from equations (32), (33) and (46),

$$SH_i^{2p+1} = 0, \quad SH_i^{2p} = \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} SA_i^{2p-\mu} SA_i^{\mu}, \qquad (i = 1, 2, \ldots, n^2-1). \tag{48}$$

We can thereby show that the non-zero roots of the characteristic equation of H_i, $|\omega - H_i| = 0$, occur in pairs of equal value but opposite sign which can be obtained by taking all possible differences of the roots of the equation $|\rho - A_i| = 0$ for $1 \leqq i \leqq n^2-1$.

From equation (44) it follows that for any linear function of the H's with constant coefficients, $H = \sum_{i=1}^{n^2-1} \alpha_i H_i$,

$$\sum_{i=1}^{n^2-1} \alpha_i E_i = \begin{pmatrix} H, & 0 \\ 0, & 0 \end{pmatrix}. \quad \text{Hence } \Bigl(\sum_{i=1}^{n^2-1} \alpha_i E_i\Bigr)^p = \begin{pmatrix} H^p, & 0 \\ 0, & 0 \end{pmatrix} \text{ and } S\Bigl(\sum_{i=1}^{n^2-1} \alpha_i E_i\Bigr)^p = SH^p$$

for any positive integer p. From this result and equations (41) we see that $SH^{2p+1} = 0$, $SH^{2p} = \sum_{\mu=0}^{2p} (-1)^{\mu} \binom{2p}{\mu} SA^{2p-\mu} SA^{\mu}$, $(p = 1, 2, \ldots)$, where $A = \sum_{i=1}^{n^2-1} \alpha_i A_i$. Therefore, the roots of $|\omega - \sum_{i=1}^{n^2-1} \alpha_i H_i| = 0$ are related to the roots of $|\rho - \sum_{i=1}^{n^2-1} \alpha_i A_i| = 0$ in the way described above. For an arbitrary choice of the α's $|\omega - \sum_{i=1}^{r} \alpha_i H_i| = 0$ is the characteristic equation of the special linear homogeneous group relative to the subgroup G_r—when $r = n^2-1$, it is the characteristic equation of the special linear homogeneous group—and $\sum_{i=1}^{r} \alpha_i A_i$ is the general infinitesimal transformation of the group. We have then the following theorem:

The non-zero roots, not exceeding $n(n-1)$ in number, of the characteristic equation of the special linear homogeneous group in n variables relative to any given subgroup occur in pairs, equal in absolute value but opposite in sign, and can be obtained by taking all possible differences of the roots of the characteristic equation of the general infinitesimal transformation of this given subgroup.

Section 3.

The above investigation of the relations between the scalar function of the products of powers of the infinitesimal transformations of the general linear homogeneous group G in n variables (or the special linear homogeneous group G' in n variables) and the adjoint of G (or G') may be applied to establish certain relations between the characteristic equation of the general infinitesimal transformation of the subgroup G_r with r parameters of G (or G') and the characteristic equation of the general infinitesimal transformation of the adjoint of G_r, termed by Lie and Killing the "characteristic equation" of this group, Killing has taken the characteristic equation of a group as the basis of his investigations upon the structure of finite continuous groups.

As in § 1, let $A_1, A_2, \ldots, A_r$ be the infinitesimal transformations of any given subgroup G_r, of the general linear homogeneous group, and $A_1, A_2, \ldots, A_r, A_{r+1}, \ldots, A_{n^2}$ the infinitesimal transformations of G; and if G_r is a subgroup of G', let $A_1, A_2, \ldots, A_r, A_{r+1}, \ldots, A_{n^2-1}$ be the infinitesimal transformations of G'. Then, see equation (3),

$$c_{ijr+h}=0, \qquad (i, j=1, 2, \ldots, r;\ h=1, 2, \ldots, n^2-r-1 \text{ or } n^2-r^*).$$

Therefore, by (4) and (5),

$$E_i=\begin{pmatrix} E'_i, & K_i \\ 0, & F_i \end{pmatrix}, \qquad (i=1, 2, \ldots, r),$$

where $(K_i)_{\nu\mu}=c_{i,\,r+\nu,\,\mu}, \qquad (\mu=1, 2, \ldots, n;\ \nu=1, 2, \ldots, n^2-r-1 \text{ or } n^2-r^*),$

$(F_i)_{\nu\mu}=c_{i,\,r+\nu,\,r+\mu}, \qquad (\mu, \nu=1, 2, \ldots, n^2-r-1 \text{ or } n^2-r^*),$

and 0 denotes a matrix with n^2-r-1 or n^2-r^* rows and r columns whose constituents are all zero.

The transformations $E_1, E_2, \ldots, E_r$ constitute the infinitesimal transformations of the adjoint of G' (or of G), relative to the subgroup G_r, and

---

*According as G_r is or is not a subgroup of G'.

$E'_1, E'_2, \ldots, E'_r$ the infinitesimal transformations of the adjoint of G_r. The general infinitesimal transformation of the adjoint of G' (or of G) relative to G_r is $\sum\limits_{i=1}^{r} \alpha_i E_i$ and the general infinitesimal transformation of the adjoint of G_r is $\sum\limits_{i=1}^{r} \alpha_i E'_i$. We have $\sum\limits_{i=1}^{r} \alpha_i E_i = \left| \begin{array}{cc} \sum\limits_{i=1}^{r} \alpha_i E'_i, & \sum\limits_{i=1}^{r} \alpha_i K_i \\ 0, & \sum\limits_{i=1}^{r} \alpha_i F_i \end{array} \right|$ and therefore,

$$\left|\omega - \sum_{i=1}^{r} \alpha_i E_i\right| = \left|\omega - \sum_{i=1}^{r} \alpha_i E'_i\right| \cdot \left|\omega - \sum_{i=1}^{r} \alpha_i F_i\right|. \tag{49}$$

The equation

$$\left|\omega - \sum_{i=1}^{r} \alpha_i E_i\right| = \omega^{\mu} - \Psi_1(\alpha)\omega^{\mu-1} + \Psi_2(\alpha)\omega^{\mu-2} - \ldots \pm \Psi_{\mu-1}(\alpha)\omega \mp \Psi_{\mu}(\alpha) = 0,$$

in which $\Psi_{\mu}(\alpha) = 0$ * is termed by Cartan the characteristic equation of G' (or of G †) relative to the subgroup G_r of G' (or of G), and the left-hand member of this equation the "characteristic determinant" of G' (or G) relative to G_r.† The equation of the general infinitesimal transformation of the adjoint of G_r,

$$\left|\omega - \sum_{i=1}^{r} \alpha_i E'_i\right| = \omega^{r} - \Psi_1(\alpha)\omega^{r-1} + \Psi_2(\alpha)\omega^{r-2} - \ldots \mp \Psi_{r-1}(\alpha)\omega \pm \Psi_{r}(\alpha) = 0,$$

in which $\Psi_r(\alpha) = 0$,‡ is termed by Lie and Killing the "characteristic equation" of the group G_r, and the left-hand member of this equation Cartan terms the "characteristic determinant" of the group G_r. The characteristics of the equation $\left|\omega - \sum\limits_{i=1}^{r} \alpha_i E'_i\right| = 0$ determine in part the structure of G_r.

Killing has shown that the coefficients of the characteristic equation $\left|\omega - \sum\limits_{i=1}^{r} \alpha_i E'_i\right| = 0$ of the group are invariants of the adjoint, and has founded a classification of groups upon the properties of these coefficients. He defines the "rank" (*Rang*) of the group as the number of the coefficients of the characteristic equation of the group that are independent (*Math. Ann.*, Vol. XXXI, p. 254). As remarked by Cartan the rank of a group is the number of roots of the characteristic equation of the group that are independent (*Thèses:* "Sur la Structure des Groupes de Transformations finis et continus," p. 28). The rank is zero if and only if the coefficients of the characteristic equation are all zero identically.

---

* See note p. 438.

† According as G_r is or is not a subgroup of G'.

‡ See note p. 438.

An important classification of finite continuous groups due to Lie consists in the division into integrable and non-integrable groups. An integrable group is a group some one of whose derived groups contains only the identical transformation ("Continuierliche Gruppen," p. 58). Finite continuous groups are also classified as "simple" and "non-simple" groups. A "simple" group is a group that contains no invariant subgroup. A group is "semi-simple" if no invariant subgroup is integrable (Cartan, *Thèses*, p. 51).

We have the following theorems due to Lie, Engel, Killing and Cartan:

(i). Groups of rank zero are integrable and the derived groups of an integrable group are of rank zero (Umlauf, *These*, Leipzig).

(ii). If the first derived of a group is integrable, then the group itself is integrable (Lie, "Continuierliche Gruppen," p. 547).

(iii). The necessary and sufficient condition that a group of order r be integrable is that the transformations of its first derived annul identically the coefficient $\Psi_2(\alpha)$ in its characteristic equation (Cartan, *Thèses*, p. 47).

(iv). If an infinitesimal transformation annuls $\Psi_{r-1}(\alpha)$ without annulling all the first minors of $\sum_{i=1}^{r} \alpha_i E_i$, it annuls all the invariants of the adjoint which are homogeneous and of degree non-zero (Cartan, *Thèses*, p. 31).

(v). The characteristic equation of a simple or semi-simple group with r parameters and of rank k has $r-k$ non-zero unequal roots which occur in pairs of equal value but opposite sign, and the transformations belonging to such a pair of roots are not commutable (Killing, *Math. Ann.*, Vol. XXXIV, p. 102).

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the roots of the characteristic equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ of the general infinitesimal transformation of G_r. It was shown in § 1 and § 2 that the roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ of G' (or G) relative to G_r are given by $\rho_i - \rho_j$ for $i, j = 1, 2, \ldots, n$. Whence it follows by (49) that the roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ of G_r are numbers contained in the aggregate of differences of the roots of the characteristic equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ of the general infinitesimal transformation of G_r.

I shall denote by l in what follows the maximum number of distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ for any infinitesimal transformation of G_r.

Let $\rho_1, \rho_2, \ldots, \rho_l$ be the distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ relative to the general infinitesimal transformation of G_r. The distinct non-zero roots of $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ corresponding to the general infinitesimal transformation of G' (or G) relative to the subgroup G_r and, therefore, the distinct non-zero roots of $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ relative to the general infinitesimal transformation of G_r are contained in the aggregate of $l^2 - l$ functions $\rho_i - \rho_j$ of the α's for $i, j = 1, 2, \ldots, l$ and $j \neq i$. But these $l^2 - l$ functions of the α's are themselves functions of the $l-1$ differences $\rho_i - \rho_l$, $(i = 1, 2, \ldots, l-1)$. Therefore, the non-zero roots of $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ relative to the general infinitesimal transformation of G_r are functions of at most $l-1$ functions of the α's, and thus at most $l-1$ of the roots of the equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ are independent functions of the α's. We have, therefore, the following theorems:

I. *If the maximum number of distinct roots of the characteristic equation of the general infinitesimal transformation of* G_r *is* l, *the rank of* G_r *can not exceed* $l-1$.

II. *If the roots of the characteristic equation of the general infinitesimal transformation of* G_r *are all equal the rank of* G_r *is zero; therefore,* G_r *is integrable.*

III. *If the roots of the characteristic equation of the general infinitesimal transformation of the first or any subsequent derived group of* G_r *are all equal, the rank of the derived group is zero; therefore,* G_r *is integrable.*

The roots of the characteristic equation of any infinitesimal transformation of the following group

$$A_1 = x_3 \frac{\partial}{\partial x_1}, \quad A_2 = x_2 \frac{\partial}{\partial x_1}, \quad A_3 = x_3 \frac{\partial}{\partial x_2}, \quad A_4 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$$

are all equal; this group illustrates theorems II and III, for the characteristic equation of the general infinitesimal transformation is

$$|\rho - \sum_{i=1}^{4} \alpha_i A_i| = (\rho - \alpha_4)^3 = 0.$$

The first derived of this group is $A_1 = x_3 \frac{\partial}{\partial x_1}$. The group contains two exceptional infinitesimal transformations, viz.,

$$A_1 = x_3 \frac{\partial}{\partial x_1}, \quad A_4 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

From what was stated above it is clear that if $l' \leqq l$ is the number of independent roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$, or the number of coefficients of this equation which are independent functions of the α's, the number of independent roots of the equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ can not exceed l'. Whence it follows that:

IV. *If l' is the number of independent coefficients of the characteristic equation*

$$|\rho - \sum_{i=1}^{r} \alpha_i A_i| = \rho^n - P_1\rho^{n-1} + \ldots \mp P_{n-1}\rho \pm P_n = 0,$$

of the general infinitesimal transformation G_r, the rank of G_r can not exceed l'.

The maximum number of distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ for any group G_r in n variables is n. Therefore, from *I* follows the theorem:

V. *The rank of no subgroup of the general linear homogeneous group in n variables can exceed $n-1$.*

By the preceding theorem the rank of the special linear homogeneous group, G', can not exceed $n-1$. Suppose the rank of G' to be $k < n-1$. G' is a simple group (*Transformationsgruppen*, Vol. I, p. 560); and hence by theorem (V) the number of non-zero roots of the characteristic equation of G' is n^2-k-1. If $k<n-1$ then $n^2-k-1>n^2-n$ or the number of non-zero roots exceeds n^2-n. But in § 2 it was shown that the number of non-zero roots does not exceed n^2-n, hence $k \nless n-1$, and we must have $k=n-1$.

Killing has shown that the number of independent invariants of the transformations of the adjoint of any group can not be less than the rank of the group (*Math. Ann.*, Vol. XXXI, p. 266). Thus the number of independent invariants of the adjoint of G' can not be less than $n-1$. Moreover, the number of invariants of the adjoint of any group G_r is the same as the nullity of the matrix $\sum_{i=1}^{r} \alpha_i E'_i$ which can not exceed the number of zero roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$. Since the rank of G' is $n-1$ it follows that the characteristic equation of G' has exactly $n-1$ roots zero and therefore the number of invariants can not exceed $n-1$ and must, therefore, be exactly $n-1$. Hence:

VI. *The rank of the special linear homogeneous group in n variables is $n-1$, and is the same as the number of independent invariants of the transformations of its adjoint group.*

From equations (44) it follows that the characteristic equation,

$$|\omega-\sum_{i=1}^{n^2}\alpha_i E_i|=\omega|\omega-\sum_{i=1}^{n_2-1}\alpha_i H_i|=0,$$

of the general linear homogeneous group, G, has the same number of independent coefficients as the characteristic equation $|\omega-\sum_{i=1}^{n^2-1}\alpha_i H_i|=0$ of G'. Thus the rank of G is equal to the rank of G' and is $n-1$. Also since the nullity of $\sum_{i=1}^{n^2-1}\alpha_i H_i$, a matrix of order n^2-1, is $n-1$ it is evident that the nullity of $\sum_{i=1}^{n^2}\alpha_i E_i=\left(\begin{array}{cc}\sum_{i=1}^{n^2-1}\alpha_i H_i, & 0\\ 0, & 0\end{array}\right)$, a matrix of order n^2, is exactly n. We have then:

VII. *The rank of the general linear homogeneous group in n variables is $n-1$ and the number of independent invariants of the transformations of its adjoint group is n.*

Let the l distinct roots of $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ be respectively of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. The equation $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ has then at least $\mu_1^2+\mu_2^2+\ldots+\mu_l^2$ zero roots. If this equation has as many as n^2-r+2 zero roots it follows from (49) that $\Psi_{r-1}(\alpha)=0$. Therefore, $\Psi_{r-1}(\alpha)=0$ if $\mu_1^2+\mu_2^2+\ldots+\mu_l^2\geqq n^2-r+2$. But Cartan has shown that, when the coefficient $\Psi_{r-1}(\alpha)$ of the characteristic equation of G_r vanishes identically, every transformation of G_r is commutative with at least one other transformation of G_r (*Thèses*, p. 32). Whence we have the following theorem:

VIII. *Let G_r be a subgroup with r parameters of the general linear homogeneous group in n variables; and let the l distinct roots of the characteristic equation of the general infinitesimal transformation of G_r be of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. If $\mu_1^2+\mu_2^2+\ldots+\mu_l^2\geqq n^2-r+2$, then every infinitesimal transformation of G_r is commutative with at least one other infinitesimal transformation of this group.*

The group

$$A_1=x_1\frac{\partial}{\partial x_1},\quad A_2=x_1\frac{\partial}{\partial x_2},\quad A_3=x_1\frac{\partial}{\partial x_3},\quad A_4=x_1\frac{\partial}{\partial x_4},\quad A_5=x_3\frac{\partial}{\partial x_2},$$

$$A_6=x_3\frac{\partial}{\partial x_4},\quad A_7=x_4\frac{\partial}{\partial x_2},\quad A_8=x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4},$$

illustrates the above theorem. The characteristic equation of the general infinitesimal transformation is $(\rho-\alpha_1)(\rho-\alpha_4)^3=0$. Thus $\mu_1^2+\mu_2^2=10=n^2-r+2$.

This group contains no exceptional transformation but $(A_1, A_8) = (A_2, A_7) = (A_3, A_4) = (A_5, A_6) = 0$. The group given on page 446 also illustrates this theorem. In this case we have $\mu_1^2 = 9 > 7 = n^2 - r + 2$, and the group contains two exceptional transformations.

By theorem (v) the characteristic equation of a simple or semi-simple group with r parameters and of rank k has $r-k$ distinct non-zero roots. I have shown that the number of distinct non-zero roots of the characteristic equation of G_r can not exceed l^2-l, and that the rank of G_r can not exceed $l-1$ or exceed l'. Suppose G is a simple group of rank k, then from Killing's theorem and what has just been stated $k \leqq l-1$, $k \leqq l'$ and $r-k \leqq l^2-l$. Therefore, $r \leqq l^2-1$ and $r \leqq l^2-l+l'$. We have then the following theorem:

IX. *Let G_r be a subgroup with r parameters of the general linear homogeneous group; let l be the maximum number of distinct roots of the characteristic equation of the general infinitesimal transformation of G_r and let l' be the number of these roots which are independent. If either l^2-1 or l^2-l+l' is less than r then the group can be neither simple nor semi-simple.*

The example given on page 448 illustrates this theorem for $l^2-1 = 3 < 8 = r$ and the group contains the invariant subgroup A_2, A_3, A_4, A_5, and by III is integrable.

Since $\Psi_r(\alpha) = |\sum_{i=1}^{r} \alpha_i E'_i|$ is identically zero, the nullity of the matrix $\sum_{i=1}^{r} \alpha_i E'_i$ is at least 1. For a given infinitesimal transformation $\sum_{i=1}^{r} \alpha_i A_i$ of G_r, let the nullity of $\sum_{i=1}^{r} \alpha_i E'_i$ be unity, and let the equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ have two roots zero. Then, for the given infinitesimal transformation $\sum_{i=1}^{r} \alpha_i A_i$ of G_r, $\Psi_{r-1}(\alpha)$ is annulled, but not all the first minors of $\sum_{i=1}^{r} \alpha_i E'_i$, and hence by theorem (iv) all the invariants of the adjoint, homogeneous and of degree non-zero, are annulled by the given infinitesimal transformation. Let $\mu_1, \mu_2, \ldots, \mu_l$ be the multiplicities of the distinct roots of the characteristic equation of $\sum_{i=1}^{r} \alpha_i A_i$. From (49) it follows that $\Psi_{r-1}(\alpha) = 0$ if $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ has at least n^2-r+2 zero roots, which will be the case if $\sum_{i=1}^{l} \mu_i^2 \geqq n^2-r+2$ (cf. p. 448); and the nullity of $\sum_{i=1}^{r} \alpha_i E'_i$ will not exceed unity if the nullity of $\sum_{i=1}^{r} \alpha_i E_i$ does not exceed n^2-r+1.

From (13) it follows that the nullity of $\sum_{i=1}^{r} \alpha_i E_i$ is equal to the nullity of $\sum_{i=1}^{r} \alpha_i (M_i - N_i)$. We have then the following theorem:

X. *Let* G_r *be a subgroup with* r *parameters of the general linear homogeneous group in* n *variables. Let the* $l \leqq n$ *distinct roots of the characteristic equation of any given infinitesimal transformation* $\sum_{i=1}^{r} \alpha_i A_i$ *of* G_r *be respectively of multiplicity* $\mu_1, \mu_2, \ldots, \mu_l$. *If* $\sum_{i=1}^{l} \mu_i^2 \geqq n^2 - r + 2$ *and the nullity of* $\sum_{i=1}^{r} \alpha_i (M_i - N_i)$ (*see p.* 437) *does not exceed* $n^2 - r + 1$, *the invariants of the adjoint, homogeneous and of degree non-zero, are then all annulled for this infinitesimal transformation.*